SOME NONLINEAR DYNAMICS OF A HEATED CHANNEL *

RIZWAN-UDDIN ** and J.J. DORNING ***
Department of Nuclear Engineering and Engineering Physics, University of Virginia, Charlottesville, VA 22901, USA

Received December 1985

Linear and nonlinear mathematical stability analyses of parallel channel density wave oscillations are reported. The two phase flow is represented by the drift flux model. A constant characteristic velocity $v_\infty$ is used to make the set of equations dimensionless to ensure the mutual independence of the dimensionless variables and parameters: the steady-state inlet velocity $\tilde{\vartheta}$, the inlet subcooling number $N_{sub}$ and the phase change number $N_{pc}$. The exact equation for the total channel pressure drop is perturbed about the steady-state for the linear and nonlinear analyses. The surface defining the marginal stability boundary (MSB) is determined in the three-dimensional equilibrium-solution/operating-parameter space $\tilde{\vartheta} - N_{sub} - N_{pc}$. The effects of the void distribution parameter $C_0$ and the drift velocity $V_g/2$ on the MSB are examined. The MSB is shown to be sensitive to the value of $C_0$ and comparison with experimental data shows that the drift flux model with $C_0 > 1$ predicts the experimental MSB and the neighboring region of stable oscillations (limit cycles) considerably better than do the homogeneous equilibrium model ($C_0 = 1, V_g/2 = 0$) or a slip flow model. The nonlinear analysis shows that supercritical Hopf bifurcation occurs for the regions of parameter space studied; hence stable oscillatory solutions exist in the linearly unstable region in the vicinity of the MSB. That is, the stable fixed point $\tilde{\vartheta}$ becomes unstable and bifurcates to a stable limit cycle as the MSB is crossed by varying $N_{sub}$ and/or $N_{pc}$.

1. Introduction

Mathematical analyses of density wave instabilities in heated channels are necessary to determine the stable operating regimes in parameter space for boiling water reactors, steam generators and other components in nuclear, chemical and petroleum industry systems. The problems that arise in analytical studies of density wave oscillations can be classified broadly into three main categories - those that arise in: (a) mathematical modeling of the two phase flow; (b) modeling of specific systems; and (c) mathematical solution of the resulting equations. (A separate problem that indirectly affects the analytical study of density wave oscillations is the limited amount of experimental data available.) There have been over simplifications in each category, and only a step by step elimination of the simplifying assumptions made in the past will lead to an adequate understanding of the salient phenomena involved, and make possible accurate predictions of experimental data and stable operating regimes of engineering system components. Since the general two fluid model is too cumbersome to treat analytically at present, all mathematical analyses of density wave oscillations done to date use either the homogeneous equilibrium model or the drift flux model for the two phase flow. Although Ishii and Zuber [1,2] used the drift flux model in their stability analysis, they assumed a flat void profile across the channel ($C_0 = 1$); moreover, they carried out a linear stability analysis only. The nonlinear analysis shows that supercritical Hopf bifurcation occurs for the regions of parameter space studied; hence stable oscillatory solutions exist in the linearly unstable region in the vicinity of the MSB. That is, the stable fixed point $\tilde{\vartheta}$ becomes unstable and bifurcates to a stable limit cycle as the MSB is crossed by varying $N_{sub}$ and/or $N_{pc}$.
and Friedly and Krishnan [5] also used the homogeneous equilibrium model. Some of these limitations have been removed in the nuclear reactor, thermal hydraulics, stability code NUFREQ-NP[6] which utilizes the drift flux model and also incorporates either point reactor kinetics or multi-dimensional reactor kinetics to couple the parallel-channel two phase flow with the fission reactor kinetics. However NUFREQ-NP only determines linear stability. It does not address the question of nonlinear stability; hence regions where stable and unstable limit cycles exist and the amplitudes and frequencies of these nonlinear oscillations are not calculated.

To be able to determine the distance from instability of operating points of a system subjected to finite perturbations, nonlinear analysis of density wave oscillations in two- and three-dimensional parameter space for realistic models is still needed. Hence, we have carried out linear and nonlinear stability analyses using the drift flux model. Different friction factors for the single phase and two phase regions were included as were inlet and outlet restrictions. We introduce a characteristic but otherwise arbitrary velocity $v^*$ when making the variables dimensionless and thereby keep the three dimensionless variables and parameters $\bar{v}$, $N_{\text{ub}}$, and $N_{\text{pc}}$ independent of each other. Extending the ideas used by Achard et al. [3] in their stability analyses using the homogeneous equilibrium model, we decouple the continuity and energy equations from the momentum equation and solve the void propagation (mixture density) equation analytically along its characteristics to obtain the velocity and density fields as functions of the inlet velocity. We use these results in the momentum equation which we then integrate along the channel length to obtain a nonlinear, functional, delay, integro-differential equation for the inlet velocity $v(t)$ in terms of two phase residence time $\tau(t)$. The necessary coupled equation, a nonlinear delay integral equation in $v(t)$ and $\tau(t)$ is obtained by evaluating the characteristic equation of the density propagation equation at the channel exit. We first linearize these equations about a steady-state and solve the linear equations for the marginal stability boundary in a three-dimensional, dimensionless equilibrium-variable/operating-parameter space (equilibrium channel inlet velocity $\bar{v}$, channel inlet subcooling number $N_{\text{ub}}$ and phase change number $N_{\text{pc}}$). We then study the Hopf bifurcation on the boundary (surface) of marginal stability using the Lindstedt-Poincaré technique. We thus determine that, for the parameter regions of interest, the Hopf bifurcation is supercritical and thereby show that there are regions in three-dimensional operating parameter space beyond the marginal stability boundary in which there exist stable nonlinear oscillations, i.e. the stable fixed points bifurcate into stable limit cycles as the MSB is crossed. The very substantial mathematical manipulations involved in our application of the Lindstedt–Poincaré technique and our calculation of the amplitudes and frequencies of the stable nonlinear oscillations have been carried out using the algebraic programming system REDUCE-2[7] on the University of Virginia CDC-180/855.

Comparisons with previous theoretical results obtained using the homogeneous equilibrium model [3] and the drift flux model with $C_0 = 1$ (a slip flow type model) [1,2] and with experimental data show that the present theoretical results are in good agreement with the data, except at low subcooling number, and thus that the use of a drift flux model with $C_0 = 1$ is important in modeling the two phase flows.

2. Model

Using the assumption of thermal equilibrium we divide the heated channel into single phase and two phase regions. In the single phase region, the fluid density is taken as constant and equal to the density of the liquid phase; thus the velocity in this region is equal to the velocity at the channel inlet (incompressible liquid phase) and the momentum conservation equation is

$$\begin{align*}
\frac{\partial p^*_\text{in}}{\partial z^*} + \rho^*_T f^*(v^*(t^*))^2 + \rho^*_T g^* \frac{d v^*(t^*)}{dt^*} + \frac{\rho^*_T}{2D^*} + \rho^*_T g^*.
\end{align*}$$

Here the (*) indicates a dimensional quantity (to be made dimensionless below). The single phase region height $\lambda^*$ is given by

$$\begin{align*}
\lambda^*(t^*) = \int_{z^*}^{x^*} v^*(t^*) dz^*.
\end{align*}$$

where $v^* = \rho^*_T A^* A^* h^*/q''^*$ is the time the incoming liquid takes to reach the saturation temperature, as it travels axially along the channel, for a given heat flux $q''^*$. The drift flux model for the two phase region is characterized by vapor drift-velocity $V_g^*$, and void distribution parameter $C_0$. The cross sectionally averaged void propagation equation is given by Zuber and Staub [8]. For the case of thermal equilibrium, constant heat flux along the channel and constant $C_0$ and $V_g^*$, the drift flux equations can be written in the form [1,8]

$$\begin{align*}
\frac{\partial j^*(z^*, t^*)}{\partial z^*} = \frac{I_{pv}^* - 2 \rho^*_T}{\rho^*_T - 1}.
\end{align*}$$

2. Model

Using the assumption of thermal equilibrium we divide the heated channel into single phase and two phase regions. In the single phase region, the fluid density is taken as constant and equal to the density of the liquid phase; thus the velocity in this region is equal to the velocity at the channel inlet (incompressible liquid phase) and the momentum conservation equation is

$$\begin{align*}
\frac{\partial p^*_\text{in}}{\partial z^*} + \rho^*_T f^*(v^*(t^*))^2 + \rho^*_T g^* \frac{d v^*(t^*)}{dt^*} + \frac{\rho^*_T}{2D^*} + \rho^*_T g^*.
\end{align*}$$

Here the (*) indicates a dimensional quantity (to be made dimensionless below). The single phase region height $\lambda^*$ is given by

$$\begin{align*}
\lambda^*(t^*) = \int_{z^*}^{x^*} v^*(t^*) dz^*.
\end{align*}$$

where $v^* = \rho^*_T A^* A^* h^*/q''^*$ is the time the incoming liquid takes to reach the saturation temperature, as it travels axially along the channel, for a given heat flux $q''^*$. The drift flux model for the two phase region is characterized by vapor drift-velocity $V_g^*$, and void distribution parameter $C_0$. The cross sectionally averaged void propagation equation is given by Zuber and Staub [8]. For the case of thermal equilibrium, constant heat flux along the channel and constant $C_0$ and $V_g^*$, the drift flux equations can be written in the form [1,8]

$$\begin{align*}
\frac{\partial j^*(z^*, t^*)}{\partial z^*} = \frac{I_{pv}^* - 2 \rho^*_T}{\rho^*_T - 1}.
\end{align*}$$
\[
\frac{\partial \alpha^*}{\partial t^*} (z^*, t^*) + \frac{\partial}{\partial z^*} \left\{ \alpha^* \left[ C_0 j^* (z^*, t^*) + \bar{V}_{\text{eff}}^* \right] \right\} = \frac{\Gamma_{\text{sat}}^*}{\rho_0^*} \tag{4}
\]

for the conservation of volumetric flux and void fraction, and
\[
\rho_m^* \left( \frac{\partial \rho_m^*}{\partial t^*} + \bar{V}_m^* \frac{\partial \rho_m^*}{\partial z^*} \right) = -\frac{\partial P_m^*}{\partial z^*} - \frac{f_m}{2D^*} \rho_m^* \bar{h}_{\text{eff}}^*^2 - g \rho_0^* \alpha^* \frac{\partial}{\partial z^*} \left( \frac{\alpha^* \rho_m^* \rho_f^*}{\rho_m^*} \bar{V}_{\text{eff}}^* \right) \tag{5}
\]

for the conservation of momentum of the mixture, where
\[
V_{\text{eff}}^* = \bar{V}_{\text{eff}}^* + (C_0 - 1) j^* (z^*, t^*),
\]
\[
\alpha^* (z^*, t^*) = \left( 1 - \frac{\rho_m^* (z^*, t^*)}{\rho_f^*} \right) \frac{\rho_f^*}{\rho_m^*},
\]
\[
\bar{V}_m^* = j^* (z^*, t^*),
\]
\[
+ \left[ \bar{V}_{\text{eff}}^* + (C_0 - 1) j^* (z^*, t^*) \right] \left( 1 - \frac{\rho_f^*}{\rho_m^* (z^*, t^*)} \right).
\]

We rewrite these equations in dimensionless variables defined in terms of the channel length \( l^* \), a characteristic but constant, typical velocity \( v_f^* \), the fluid density \( \rho_f^* \), and the latent heat \( \Delta h_{\text{sat}}^* \). Although in this work we choose the characteristic velocity \( v_f^* \) to be a typical inlet velocity in each of the experiments with which we compare our results, it could be defined in other ways that might be convenient. For example, it could be defined in terms of a design parameter \( l^* \) as \( v_f^* = \sqrt{g^* l^*} \), in which case the Froude number would become unity and there would be one fewer dimensionless parameters, as is the case in refs. [1-3]. This still would result in a set of dimensionless equilibrium variable and operating parameters \( \bar{\delta}, N_{\text{sub}}, \) and \( N_{\text{pch}} \) that would have the same properties as those actually used here: mutual independence and a one-to-one correspondence with the independently variable, dimensional, experimental, operating quantities \( v_f^*, \Delta h_{\text{sat}}^* \), and \( q^* \). However, with this definition, a comparison with the results of Ishii and Zubler [1,2] and Saha et al. [13] in the \( N_{\text{sub}}-N_{\text{pch}} \) plane would require a linear stretching of our results parallel to the \( N_{\text{pch}} \) axis. The dimensionless variables and parameters used here are given in Appendix A. The resulting dimensionless equations are
\[
\rho = 1 \quad (0 < z < \lambda), \quad j(t) = v(t) - v_i(t), \quad \lambda (t) = \int_{l^*}^z v(t') \, dt',
\]
\[
\frac{\partial P_{1e}}{\partial z} = \frac{d \rho_m^* (t)}{dt} + N_i v_i^2(t) + Fr^{-1}. \tag{6}
\]

for the single phase region, and
\[
\frac{\partial \rho_m^* (z, t)}{\partial z} = N_{\text{pch}}, \quad \frac{\partial P_{2e}}{\partial z} = (C_0 j^* (z, t) + \bar{V}_{\text{eff}}^*) \frac{\partial \rho_m^* (z, t)}{\partial z}
\]
\[
= -N_{\text{pch}} [C_0 (\rho_m^* (z, t) - 1) + 1], \quad \frac{\partial P_{2e}}{\partial t} + v_m \frac{\partial \rho_m^* (z, t)}{\partial z} + Fr^{-1}
\]
\[
+ N_i v_i^2 (z, t) + \frac{\partial}{\partial z} \left( \frac{\alpha^* \rho_f^*}{1 - \alpha^* \rho_m^* (z, t)} \right) \tag{10},
\]
\[
\alpha^* (z, t) = \left( 1 - \rho_m^* (z, t) \right) N_i,
\]
\[
\rho_m^* (z, t) = \left( \frac{v_i (z, t)}{1 - \rho_m^* (z, t)} \right) \tag{11},
\]
\[
\text{for the two phase region.}
\]

3. Analysis

In order to study the linear and nonlinear stability of two phase flow in a heated channel using the drift flux equations we used characteristics methods to reduce the nonlinear partial differential equations in space and time to two coupled equations in time only, a nonlinear ordinary differential equation and a nonlinear integral equation. This is a direct, but nontrivial extension of work that was done earlier for the homogeneous equilibrium model equations [3]. The equation which describes the density wave oscillations is the equation for the total pressure drop in the channel
\[
\Delta P_{\text{ex}} (t) = \Delta P_e (t) + \Delta P_{1e} (t) + \Delta P_{2e} (t) + \Delta P_{1o} (t), \tag{11}
\]
\[
\text{where} \Delta P_{1e} \text{ is the imposed, external pressure drop, and the inlet pressure drop} \Delta P_e, \text{ the exit pressure drop} \Delta P_e \text{ and the single phase region pressure drop} \Delta P_{1e} \text{ are given by}
\]
\[
\Delta P_{1e} (t) = k e v_i^2 (t), \quad \Delta P_e (t) = k e \rho_m^* (z = 1, t) v_i^2 (z = 1, t), \quad \Delta P_{1o} (t) = \left( \frac{d v_i (t)}{dt} + N_i v_i^2 (t) + Fr^{-1} \right) \lambda (t). \tag{14}
\]

To evaluate the two phase region pressure drop \( \Delta P_{2e} \) we first evaluate \( j(z, t), \rho_m(z, t) \) and \( v_m(z, t) \). In-
Integrating eq. (8) from the boiling boundary to \( z \) and using the boundary condition that \( j(z = \lambda, t) \) is equal to the channel inlet velocity \( v_i(t) \), we obtain

\[
j(z, t) = v_i(t) + N_{p_{ch}}(z - \lambda).
\]

Substituting the above expression for \( j(z, t) \) into eq. (9) and integrating the resulting equation along its characteristics using the initial condition \( z = \lambda(t) \) at \( t = t_0 \), where \( t_0 \) is the time at which the frames moving with the characteristic velocity, \( C_0[v_i(t) + N_{p_{ch}}(z - \lambda)] + \frac{V_e}{N_{p_{ch}}} \), leave the boiling boundary, yields

\[
z - \lambda(t) = \int_{t_0}^{t} e^{N_{p_{ch}}(t - t')} v_i(t' - \nu) \, dt'.
\]

(16)

and

\[
\rho_m(t - t_0) = e^{-N_{p_{ch}}(t - t_0)} + \left( \frac{C_0 - 1}{C_0} \right) \left( 1 - e^{-N_{p_{ch}}(t - t_0)} \right),
\]

(17)

where \( N_{p_{ch}} = N_{p_{ch}} \cdot C_0 \). Eq. (16) can be used to define a "two phase residence time" \( \tau(t) \), by evaluating it at \( z = 1 \) (the channel exit),

\[
1 = \lambda(t) + \int_{t_0}^{t} e^{N_{p_{ch}}(t' - t'' - \nu)} \, dt''.
\]

(18)

Now the two phase region pressure drop in the channel can be calculated by integrating eq. (10) from \( z = \lambda \) to \( z = 1 \). This integration is simplified by changing the integration variable from \( dz \) to \( dt' \), where \( t' = t - t_0 \). The result is

\[
\Delta P_{ex}(t) = \int_{t_0}^{t} \rho_m(t') \left[ \frac{\partial v_i(t')}{\partial t} - v_i(t') + \frac{\partial v_m}{\partial z} + Fr^{-1} + N_{f_{ch}} \right] \times e^{N_{p_{ch}}(t' - t - \nu)} \, dt'
\]

\[
+ \left( \frac{C_0 - 1}{C_0} \right) v_i(t' - \nu) \, dt' + N_{f_{ch}} \left[ \frac{\alpha(z = 1, t)}{1 - \alpha(z = 1, t)} \frac{v_i^2(z = 1, t)}{\rho_m(z = 1, t)} \right].
\]

(19)

where the upper integration limit \( \tau(t) \) is the two phase residence time given by eq. (18). Substituting eqs. (12) (14) and (19) into eq. (11) yields a nonlinear, functional, first order, ordinary differential equation for the channel inlet velocity in terms of the two phase residence time

\[
\Delta P_{ex}(t) = \left( k_i v_i^2(t) + k_r \rho_m(\tau) v_m^2(\tau) \right)
\]

\[
+ \left[ \frac{\partial v_i(t)}{\partial t} + N_{f_{ch}} v_i^2(t) + Fr^{-1} \right] \lambda(t)
\]

\[
+ \left[ \frac{\partial v_m(t)}{\partial t} + \frac{N_{p_{ch}}}{v_i(t - \nu)} \right] + N_{p_{ch}} \left[ v_i(t - \nu) \right]
\]

(20)

where the \( \lambda(t) \), which include complicated, nonlinear, multiple delay integrals are given in Appendix B. Eqs. (18) and (20) for the two-phase residence time \( \tau(t) \), and the channel inlet velocity \( v_i(t) \) give the nonlinear dynamical behavior of a heated channel, in which the two phase flow is described by the drift flux model, due to a pressure drop \( \Delta P_{ex} \) (which in general can be time dependent) imposed across it and a constant heat flux along its length. These equations can be recast in a fairly general form as a set of coupled, first order, nonlinear, functional, ordinary differential equations describing the heated channel as a nonlinear dynamical system. This form is obtained directly by differentiating eq. (18) with respect to time and replacing this integral equation by the first order ordinary differential equation that results. For a constant, steady-state, inlet velocity \( \bar{v} \) eqs. (18) and (20) relate the imposed \( \Delta P_{ex} \) and the steady-state inlet velocity. The evaluation of some of the integrals given in Appendix B required the expansion of \( 1/\rho_m \) (\( \rho_m \) is given by eq. (17)) in powers
of \((C_0 - 1)\). In this study we have neglected the second and higher order terms in this quantity. The resulting restriction on \(C_0\) is \(C_0 - 1 \ll \exp[-Npch\tau]\).

For the quality at the channel exit to be between zero and one, \(\bar{\delta}\) must be between finite maximum and minimum values. Simple conservation of energy leads to expressions for these quantities as functions of \(N_{sub}\) and \(N_{pch}\)

\[
\delta_{max} = \frac{N_{pch}}{N_{sub}},
\]

\[
\delta_{min} = \frac{N_{pch}N_{sub}}{1 + N_{sub}N_{pch}}.
\]

Hence for the present analysis to be relevant, the operating points must be between the surfaces \(\bar{\delta}_{max}\) and \(\bar{\delta}_{min}\) defined by these equations and shown in fig. 1 for water at a pressure of 1000 psia.

For the case of a single channel in parallel with a large bypass \(\Delta P_{ex}\) can be assumed constant; thus, perturbing eqs. (18) and (20) about the steady state and keeping terms through the third order yields two equations of the form

\[
L_1(\delta v; \xi) + Q_1(\delta v, \delta v; \xi) + C_1(\delta v, \delta v, \delta v; \xi) = 0,
\]

\[
L_2(\delta v; \xi) + Q_2(\delta v, \delta v; \xi) + C_2(\delta v, \delta v, \delta v; \xi) = 0,
\]

comprised of terms linear \((L)\), quadratic \((Q)\) and cubic \((C)\) in the perturbed unknowns. Each term in the above equations also depends upon the parameters in eqs. (20) and (18) represented symbolically here as the vector of parameters \(\xi\). The second of the above equations, obtained by perturbing eq. (18), is an algebraic equation, in \(\delta\), thus in the linear analysis \(\delta\) can be obtained explicitly in terms of \(\delta v\) and substituted in the linear form of the first equation, reducing the system of two linear differential equations to one ordinary differential equation for \(\delta v\). Similarly, in the nonlinear analysis in which the perturbed unknowns \(\delta v\) and \(\delta\) are developed in asymptotic expansions, the linear form of the above coupled equations that arises in general in the \(n\)th order of the hierarchy can be reduced to one linear ordinary differential equation in the unknown \(n\)th order velocity in the asymptotic expansion.

### 4. Linear analysis

Keeping only the linear terms in eqs. (21), and taking the Laplace transform gives the transformed solution \(\bar{\delta} v\) in the form

\[
\bar{\delta} v(s) = \frac{F(s)}{\phi(s)},
\]

where the characteristic equation is of the form

\[
\phi(s) = \frac{1}{s(N_{pch}^\prime - s)(2N_{pch}^\prime - s)(3N_{pch}^\prime - s)} \times \left(\phi_1(s) + \phi_2(s)e^{-s\tau} + \phi_3(s)e^{-2s\tau} + \phi_4(s)e^{-3s\tau}\right)
\]

and the \(\phi_i(s)\) are polynomials in \(s\) up to sixth degree.

We determine the MSB (corresponding to the first pair of complex conjugate roots of \(\phi(s)\) crossing the imaginary axis) in the \(\bar{\delta}-N_{sub}-N_{pch}\) dimensionless, equilibrium-variable/operating-parameter space by setting \(\text{Re}(s) = 0\) in eq. (22) and solving for two of these three quantities with the third as well as all other parameters fixed and then varying the third, etc. The resulting MSB surface, separating the linearly stable region \(D_1\) below it from the linearly unstable region \(D_2\) above it, is shown viewed at two different angles in figs. 2a and 2b. This complete surface lies above the \(\bar{\delta}_{min}\) surface and below the \(\bar{\delta}_{max}\) surface shown in fig. 1. Two-dimensional projections of the MSB surface onto the \(\bar{\delta}-N_{sub}\), \(N_{sub}-N_{pch}\) and \(N_{sub}-\bar{\delta}\) planes yield linear stability boundary curves that are almost straight lines in the \(\bar{\delta}-N_{pch}\) plane and \(s\)-shaped curves in both the \(N_{sub}-N_{pch}\) and \(N_{sub}-\bar{\delta}\) planes.
5. Nonlinear analysis

Since the nonlinear analysis for the dynamic behavior of the heated channel most likely cannot be carried out exactly, asymptotic expansion methods are used here to obtain oscillatory solutions in the vicinity of the MSB for the nonlinear problem. This is based on the fact that limit cycles are observed experimentally in heated channels [9,10]. The Hopf theory [11,12] is used to establish the existence and nature of periodic solutions for nonlinear differential equations. The details of the application of this theory involve starting from a point on the MSB and perturbing about it. There are two possible results of such a calculation: "supercritical" Hopf bifurcation in which there exist stable oscillatory solutions (stable limit cycles) for operating points in a strip beyond the MSB (i.e., in the linearly unstable region), and "subcritical" Hopf bifurcation in which there exist unstable oscillatory solutions (unstable limit cycles) for operating points in a strip inside the MSB (i.e., in the linearly stable region). The analysis used in studying these two possibilities is based on the Floquet theory [11,12] of linear differential equations with periodic coefficients. The nonlinear, functional, ordinary differential equation, eq. (20) and the nonlinear integral equation, eq. (18) that we have developed here for two phase flow in heated channels satisfy the conditions for general Hopf bifurcation for retarded functional differential equations [12]: namely, with all other roots with strictly negative real parts, a pair of isolated complex conjugate roots of the characteristic equation of the linearized problem crosses the imaginary axis with finite velocity as the bifurcation parameter is varied across the MSB.

We use the Lindstedt–Poincaré perturbation technique [12] to determine the nonlinear "modifications" of the amplitude and frequency of the oscillatory solutions for a point off the MSB, to those of the linear theory solution for a point on the MSB. To do this we expand the dependent and independent variables in the small parameter \( \epsilon \) as

\[
\begin{align*}
\phi(t) &= \bar{\phi} + \delta \phi(t), \\
\phi_0(t) &= \bar{\phi}_0 + \delta \phi_0(t), \\
\phi_1(t) &= \phi_0 + \delta \phi_1(t)
\end{align*}
\]

where \( \omega_0 = \omega_0(N_{\text{veh}}, N_{\text{pbr}}, \bar{c}, \ldots) \) is the magnitude of the pure imaginary part of \( \phi(i\omega) = 0 \).

The solution to the linearized equations for the inlet velocity \( \phi_1(t) \) when the operating point is on the MSB, i.e., on the MSB surface in the three-parameter space shown in figs. 2a and 2b, is given by the equilibrium solution \( \bar{v} = \bar{v}(N_{\text{veh}}, N_{\text{pbr}}, \ldots) \) plus a stable oscillatory solution with amplitude equal to the perturbation

\[
\begin{align*}
\delta v_0(\theta) &= \epsilon v_0(\theta) + \epsilon^2 v_1(\theta) + \ldots, \\
\delta v_1(\theta) &= \epsilon v_0(\theta) + \epsilon^2 v_1(\theta) + \ldots
\end{align*}
\]

where \( \omega_0 \) is the absolute value of the imaginary part of the complex conjugate roots of the characteristic equation, eq. (22), which have a zero real part on the MSB. It is introduced so that these roots will be \( \pm i \) when the operating point is on the MSB. The small parameter \( \epsilon \) is defined by

\[
\mu = \epsilon \mu_1 + \epsilon^2 \mu_2 + \ldots
\]

Here \( \xi \) and \( \mu \) are unknown constants to be evaluated by invoking the conditions for periodic solutions (i.e., eliminating secular solutions) for all \( v_n \), and \( \mu \) is the
horizontal (or vertical) distance from the MSB to the operating point in a two-dimensional parameter space, i.e.,
\[ \mu = N - N_0, \]
where \( N \) is the value of one of the parameters at the operating point and \( N_0 \) is its value on the MSB. Thus \( \mu \) is the distance parallel to one of the parameter axes from the operating point to the MSB. In short, one of the parameters is expanded about its value on the MSB to define the small parameter which then is used to expand the solution at the operating point about the solution on the MSB. Substituting eqs. (23) through (26) into eq. (21) and setting the coefficients of the various powers of \( \epsilon \) separately equal to zero yields
\[
L(v_0) = 0,
\]
\[
L(v_1) = S_1(v_0, \xi_1, \mu_1; \delta),
\]
\[
L(v_2) = S_2(v_0, v_1, \xi_1, \mu_1, \xi_2, \mu_2; \delta),
\]
after eliminating the \( \tau_0(\theta) \) in each order of the hierarchy. Next these equations are solved sequentially and \( \xi_i \) and \( \mu_i \) are calculated using the Fredholm alternative to eliminate the secular terms that would otherwise arise as a result of the forcing terms that appear on the right hand sides of the last two equations. Finally, \( \epsilon \) which determines the amplitude of the stable oscillatory solution and the corresponding operating point can be calculated using Eq. (26) truncated at two terms. Once the asymptotic expansion has been obtained for \( v_i(t) \), all the other dependent variables, including those that depend upon both \( z \) and \( t \), can be determined by simply solving eqs. (6) through (10) in terms of the known solution for \( v_i(t) \).

Any parameter can be used to define the small quantity \( \mu \). In this analysis for simplicity we use \( N_{t_1} \) (single phase friction number), a design parameter to define \( \mu \) as
\[ \mu = \bar{N}_{t_1} - N_{t_10}, \]
and initially solve the problem in the \( N_{t_1}-N_{sub} \) space, a two-dimensional design-parameter/operating-parameter space. Once the subcritical or supercritical Hopf bifurcation regions are determined, we can transform them to any other desired parameter space, since \( \mu' \), defined as \( \mu' = \bar{N}_{sub} - N_{sub} \), can be determined from the bifurcation points in the \( N_{t_1}-N_{sub} \) plane. (See fig. 3.) In another parameter space \( N_{sub} - N_{t_1} \), defined for a fixed value of \( N_{t_1} = \bar{N}_{t_1} \), where \( N_{sub} \) and \( N_{t_1} \) can both be operating parameters, the same subcritical or supercritical bifurcation points will be located at \( (\bar{N}_{sub}, \bar{N}_{t_1}) \) in the \( N_{sub}-N_{t_1} \) two-dimensional operating parameter space. Hence, we are able to calculate the Hopf bifurcation phenomena in the mathematically most convenient parameter space, based on the simplicity of the manner in which the parameters appear in eqs. (18) and (20), and still present the results in physically meaningful, operating parameter spaces.

6. Results and comparison with experiment

MSB curves have been generated in the two-dimensional operating-parameter (\( N_{sub}-N_{p_{ch}} \)) plane using eq. (22) for various fixed values of the other parameters with the two phase friction factor \( f_m = 2f_s \). These MSB curves shown in fig. 4 through 7, are compared with the experimental MSB of ref. [15]. The experimental data points for the MSB given in ref. [15] in terms of the inlet liquid temperature and the total power supplied to the channel are not consistent with the points for the MSB given in refs. [13] and [15] in terms of the dimensionless numbers \( N_{sub} \) and \( N_{p_{ch}} \). (Apparently this is due to errors in thermodynamic properties that were used to calculate the dimensionless numbers in refs. [13] and [15].) We have recalculated these dimensionless numbers and use the new values in this paper. (In a recent conference paper [17] in which we reported many of the present results, we used the actual values of the data points from ref. [13], not realizing that they were plotted there using incorrect dimensionless parameters as coordinates.) The effect of steady-state inlet velocity was studied by varying \( \delta \). This was easy to do here because the dimensionless
Fig. 4. The effect of inlet velocity $v^*$ on the marginal stability boundary and comparison with the experimental results of Saha, Ishii and Zuber [13]. The experimental data points are for set V and VI. (Since the dimensionless form of the experimental data points as reported in ref. [15] are not consistent with the dimensional form reported there, the experimental data points in $N_{ph}$, $N_{sub}$ space used in figs. 4 through 7 have been recalculated from the data reported in ref. [15] in terms of dimensional inlet temperature ($T_i$) and power supplied to the channel ($\dot{Q}$).

Fig. 5. The effect of void distribution parameter $C_0$ on the marginal stability boundary. The experimental data points are for set VI from ref. [13].

The parameters $F_r$ and $N_{ph}$ have been defined in such a way that they do not depend upon $F^*$, as they did in previous studies. The calculated MSB curves for two values of $F^*$ are shown in fig. 4 along with the experimental data of Saha et al. [13], and the analytical prediction of the MSB by Ishii and Zuber's simplified criterion [1], and by the thermal nonequilibrium theory of Saha and Zuber [14] which includes a subcooled boiling model. The MSB generated using Ishii and Zuber's [1] simplified criterion is almost independent of $F^*$ (depends upon it only through the friction number); and although the nonequilibrium theory [14] MSB does depend upon $F^*$, it is very insensitive to it. The present MSB curves agree better than previous theoretical results with the experimental data and correctly predict the effect of the steady-state inlet velocity on the MSB. Two different values of $C_0$ were used with the two different inlet velocities here, because for smaller $\theta$ the length of the annular flow regime, in which $C_0 \rightarrow 1$ increases thus lowering the effective value of $C_0$ for the total two phase region. The effect of the void distribution parameter $C_0$ also was studied and the results are shown in fig. 5. It is clear that $C_0 > 1$ leads to considerably better agreement with the experimental data than does $C_0 = 1.0$ [1]. The effect of $C_0 > 1$ on the MSB is found to be stabilizing, though, the sensitivity of the MSB to $C_0$ is dependent upon its position in parameter space. The MSB is more sensitive to the values of $C_0$ for systems with higher exit qualities. Our studies of the effect of varying the value of the drift velocity on the MSB showed that it was insensitive to $V_g^*$. In fig. 6 we compare our calculated MSB, Ishii's simplified stability criterion [1] and Saha and Zuber's thermal nonequilibrium model MSB [14] with the experimental data. The present MSB is in better agreement with the data for large values of $N_{sub}$ than the earlier calculated curves [13]. For small values of $N_{sub}$ the present MSB calculated with $C_0 > 1$ is in better agreement with the data than that based on Ishii's thermal equilibrium drift flux model analysis with $C_0 = 1$; however, it agrees less well with the data than does the MSB calculated in Saha and Zuber's nonequilibrium analysis [14]. This suggests that a subcooled boiling model may be important for low values of $N_{sub}$. On the other hand the MSB calculated numerically by Dykhuizen et al. [18] in their linear analysis for a two-fluid model which, naturally, includes subcooled boiling effects also exhibits some discrepancy with the experimental data for low $N_{sub}$. The MSB calculated numerically by Dykhuizen et al. [18] is shown in fig. 7. Since it is based upon a two-fluid model it includes the effects of thermal nonequilibrium (subcooled boiling) as well as unequal velocities, and a flow regime map. Moreover, their model also included heater wall dynamics. Also shown in fig. 7 are the experimental data due to Saha et al. [13] and the present analytically determined MSB. There is good agreement between the data points and both MSB's at large values of $N_{sub}$. At low values of $N_{sub}$, although...
the numerically calculated two fluid model MSB [18] is closer to the data points than is the present theoretically obtained MSB, the discrepancy is less by only a little more than a factor of two and is substantially larger than the discrepancy between the data points and both MSB's for large values of \( N_{\text{sub}} \). A possible partial explanation for the discrepancy between the theoretical and experimental results for low \( N_{\text{sub}} \) might be based on the results of the present nonlinear analysis.

Our nonlinear stability analysis leads to a supercritical Hopf bifurcation, and thus to finite amplitude stable oscillations (limit cycles) for operating points in the linearly unstable region, in a strip adjacent to the MSB, in which the amplitude of the stable oscillations increases as the operating point is moved away from the MSB. This is in qualitative agreement with the experimental observation of Saha et al. [13], however, not enough data has been reported there to make a quantitative comparison possible. The solution for the inlet velocity in the supercritical strip is

\[
v_i(\theta) = \bar{v} + \epsilon \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) + 2\epsilon^2 \left[ \frac{a + \bar{a}}{2} \right] + \left( \frac{be^{i2\theta} + \bar{b}e^{-i2\theta}}{2} \right),
\]

where the term in the square brackets arises from the nonsecular particular solutions to the second order equation, eq. (28). The values of \( \mu_1 = 0 \) and \( \mu_2 \) (extremely complicated) have been calculated along with \( \xi_1 \) (\( = 0 \)) and \( \xi_2 \) (\( > 0 \)) [19]. This value of \( \mu_2 \) along with a fixed, small value for \( \epsilon \) defines a family of points in the supercritical strip to which there correspond stable oscillations (limit cycles) with amplitude \( \epsilon \), eq. (30). Contours of constant \( \epsilon \), assumed to be within the boundary of the supercritical Hopf bifurcation strip, for two different values of \( \epsilon \), 0.06 and 0.08, along with the present MSB and the experimental data points are shown in fig. 7. For \( N_{\text{pch}} > 5 \) and \( N_{\text{sub}} \) between ap-pro-
appropriately values, there exist stable periodic solutions (limit cycles), e.g. for \( N_{\text{ph}} = 6 \) and \( N_{\text{ub1}} < N_{\text{ub}} < N_{\text{ub2}} \) as shown in fig. 7 (linearly unstable region, but close to the MSB) there exist stable periodic solutions of amplitude less than or equal to 0.08. This is shown in greater detail in figs. 8 and 9. Fig. 8 shows the values of the parameters \( \Delta P_{\text{tot}} \) and \( N_{\text{ub}} \) for \( N_{\text{ph}} = 6 \) for the equilibrium (fixed point) \( \bar{v} = 1.0 \) which is unstable between \( C_1 \) and \( C_2 \). For values of \( C \) between these points but close to them there exist two parts of the supercritical strip; hence stable periodic oscillations of the phase variables (limit cycles) occur. Since both \( \Delta P_{\text{tot}} \) and \( N_{\text{ub}} \) must vary to satisfy the fixed point condition \( v_i = 1.0 \), neither can be used directly as the bifurcation parameter to construct a standard bifurcation diagram. Hence, we use the arc length \( C \) from this figure, measured from a point arbitrarily defined as \( C = 0 \), as the parameter in the three dimensional bifurcation diagram shown in fig.

Fig. 9. Bifurcation diagram in \( C - \delta v(t) - \delta \lambda(t) \) space, where \( C \), the bifurcation parameter, is the arc length of the curve of fixed points in fig. 8 (measured from a point arbitrarily defined as 0). At \( C = C_1 \) (for \( C > C_1 \)) and \( C = C_2 \) (for \( C < C_2 \)) the stable fixed points bifurcate into unstable fixed points and stable limit cycles. The four spiral trajectories schematically represent from left to right typical flows of initial conditions toward a stable fixed point, outward toward a stable limit cycle, inward toward a stable limit cycle and toward a stable fixed point for various values of \( C \).

Fig. 10. Portraits of the projections of phase space trajectories onto two-dimensional phase planes for set 1 [13], at \( N_{\text{ph}} = 6 \) and \( N_{\text{ub1}} = N_{\text{ub2}} \) (see fig. 7).
9. It schematically illustrates the bifurcations of the stable fixed point to stable limit cycles at \( C_1 \) and \( C_2 \). Between \( C_1 \) and \( C_2 \) the fixed point \( \delta \) is unstable. For \( C \geq C_1 \) and \( C \leq C_2 \) the limit cycles are stable. The two-dimensional limit cycles shown in this figure are the projections of the total phase portrait onto the two dimensional \( \delta v - \delta \lambda \) plane. Explicit calculated examples of the projections of limit cycles on to the various phase planes are shown in fig. 10.

For large values of \( N_{sub} \) (i.e., in the region where the calculated MSB agrees well with the experimental data points) the \( \epsilon = \text{constant contours almost coincide with the MSB and therefore with the data points, whereas for small } N_{sub}, \text{ they lie considerably above the MSB and therefore closer than it to the data points. Similarly, the MSB curve obtained numerically by Dykhuizen et al. [18] also agrees with the experimental data for large } N_{sub} \text{ much better than it does for low } N_{sub}, \text{ where the experimental data points lie in their computed linearly unstable region. Combined with these results, the present results raise the question of whether the experimental data points might correspond, not to the MSB but rather to points in the supercritical strip for which finite-amplitude, stable oscillatory solutions exist. The difficulties, briefly mentioned below, in experimentally determining the MSB for low values of } N_{sub} (\leq 2) \text{ combined with the present results motivates this suggestion. This question can only be answered by experiments in which the amplitudes of stable oscillations are measured as a function of the distance from the MSB, e.g. as } N_{pch} \text{ is increased by increasing the channel heat flux } q' \text{ while keeping } N_{sub} \text{ fixed. Extrapolations back to zero amplitude would then experimentally locate points on the MSB for the various fixed values of } N_{sub}.

7. Discussion

The present results demonstrate the importance of the void distribution parameter \( (C_0 \neq 1) \), and show the value of a nonlinear stability analysis. The utilization of a constant characteristic velocity \( v_p^* \) in the definitions of the phase charge number \( N_{pch} \), the dimensionless inlet velocity \( \delta \) and the Froude number Fr also has made convenient both the study of the variation of the MSB with inlet velocity and the comparison of the MSB’s with experimental data sets for various inlet velocities. Values of the void distribution parameter \( C_0 \geq 1 \) clearly led to better agreement with experiment than did \( C_0 = 1 \), especially for large \( N_{sub} \). The nonlinear analysis raised the possibility that the experimental data points might correspond to stable operating points in the supercritical Hopf bifurcation strip and not to points on the MSB. This would contribute to the explanation of the remaining discrepancy between the theoretical MSB calculated here and the experimental data points for low \( N_{sub} \). This suggestion is strengthened by the fact that the accurate experimental determination of the MSB is sometimes difficult, and as shown in fig. 3 of ref. [13], the experimentally determined MSB is relatively less accurate for low values of \( N_{sub} \), because of the less well defined break in the oscillation amplitudes curve, which rises more slowly for \( N_{sub} = 2 \) as the heat flux is increased. Hence, error bars, if they could be reported with the experimental data, would be largest in this region in \( N_{sub} - N_{pch} \) parameter space where the discrepancy between experimental data and theoretical prediction also is largest. Naturally, this suggestion is not meant to imply that the importance of other possible causes for the remaining discrepancy between the present theoretical results and experiment, such as the use of constant drift velocity and void distribution parameter along the two phase channel length or the use of the drift flux model itself, should be minimized. At low values of \( N_{sub} \) there are a number of different flow regimes in the two phase region and thus constant values for \( C_0 \) and \( V_{ch} \) for the whole region do not accurately represent all the existing flow regimes.

There are two other possible contributing factors to the discrepancy between the MSB calculated here and the data points for low \( N_{sub} \), that are suggested by the results of refs. [14] and [18]. These are due to the fact that the model used in the present work does not include subcooled boiling or heater wall dynamics. Both of the above mentioned analyses [14,18], in which the MSB’s obtained disagree less with the experimental data at low \( N_{sub} \) than does the MSB obtained here, include the effect of thermal nonequilibrium, and the latter also includes the heater wall dynamics. An accurate representation of subcooled boiling (the effect of thermal nonequilibrium) becomes more important as the fairly accurately represented single phase region becomes smaller with decreasing \( N_{sub} \). Although Saha and Zuber’s thermal nonequilibrium model [14] leads to a MSB that is in poor agreement with the experimental data for high \( N_{sub} \), it agrees better with the data for low \( N_{sub} \) than does the MSB calculated here. This combination suggests both the inadequacy of their \( C_0 = 1 \) void distribution parameter and the importance of their use of a subcooled boiling model. (The fact that Ishii and Zuber’s [1] drift flux model results with \( C_0 = 1 \) agree better with the experimental data for high \( N_{sub} \) than do those of Saha and Zuber’s [14] might be due to a fortuitous cancellation of the errors that arise as a result of the absence of a subcooled boiling region and a lower than
appropriate value for $C_0$ in the two phase region.) The inclusion of heater wall dynamics also may be important because the thermal inertia of the heater wall will affect stability in general [20]. Hence, perhaps the better agreement with the experimental data at low $N_{sub}$ of the MSB obtained by Dykhuizen et al. [18] than that of the MSB obtained here can be attributed in part to the inclusion in their model of the effects of thermal nonequilibrium and in part also to the inclusion of heater wall dynamics. However, although the discrepancy between the MSB calculated by Dykhuizen et al. [18] and the experimental MSB is less than the present discrepancy, it is not insignificant compared to the latter. This remaining discrepancy may be due to the inadequacy of the constitutive relationships and/or the simple flow regime map used in that work [18]. Thus, it is possible that, the experimental data points for low $N_{sub}$ might indeed lie on the true MSB which also would be predicted correctly if a model including subcooled boiling, heater wall dynamics and more accurate constitutive relationships could be used in the theoretical analysis.

8. Summary and conclusion

Linear and nonlinear stability analyses of density wave oscillations have been done using the drift flux model for the two phase flow. It has been found that using a constant characteristic velocity (rather than the steady-state dimensional inlet velocity) leads to convenient definitions of the dimensionless parameters, since this makes it simpler to study the effect of the inlet velocity on the MSB and thereby makes direct comparison with experimental data easy [13] (see fig. 4).

Use of the drift flux model including the void distribution parameter $C_0 \neq 1$, appears to be important for thermal equilibrium analysis since it leads to a MSB that agrees considerably better with the experimental data than those obtained using the homogeneous equilibrium model or the drift flux model with $C_0 = 1$ (i.e. a slip flow type model). On the other hand, the results of our thermal equilibrium analysis (with $C_0 > 1$) agree better with the experimental data than those of the thermal nonequilibrium analysis (with $C_0 = 1$) [13] only for large values of $N_{sub}$ which suggests that the inclusion of a model for the subcooled boiling region may be important for low inlet subcooling number (see fig. 6). An alternative, or more likely, supplementary explanation of this discrepancy between the experimental data points and our MSB, follows from our nonlinear analysis which leads to a supercritical Hopf bifurcation from a stable fixed point to a stable limit cycle, and thus to stable, density-wave oscillations for operating points in the linearly unstable region in a strip adjacent to the MSB. Hence, experimentally observed finite amplitude stable oscillations do not correspond to the points on the MSB; rather, they are points in the supercritical Hopf bifurcation strip in parameter space. Further experiments, in which the amplitudes and frequencies of the stable oscillations are measured as a function of the distance into the strip of nonlinearly stable operating points, would be very valuable for verification or refutation of the adequacy of the two-phase flow model used to obtain the present theoretical results (see fig. 7).

The implications of nonlinear analysis and the effects of thermal nonequilibrium and heater wall dynamics have not been included in any single theoretical study, and since it appears that all three improve the agreement with the experimental data at low $N_{sub}$, we conclude that a nonlinear analysis using the drift flux model with drift velocity $\bar{V}_{d,j} \neq 0$ and void distribution parameter $C_0 \neq 1$, that includes the effect of thermal nonequilibrium and heater wall dynamics will be a necessary step to understand the dynamics of a heated channel at low $N_{sub}$. Further, a model that includes an axially variable $C_0$, and possibly also an axially variable $V_{ij}$, may even be necessary to adequately represent systems with very low inlet subcooling number due to the large number of different two phase flow regimes present in such a channel.

Appendix A. The dimensionless variables and parameters in the single phase and drift flux equations

To make the single phase region equations and the drift flux model equations dimensionless we introduce the following dimensionless variables and parameters:

\[ \nu_{\infty} = \frac{v_{\infty}^*}{v_0^*}, \quad \nu_l = \frac{v_l^*}{v_0^*}, \quad j = \frac{j^*}{v_0^*}, \]
\[ V_{d,j} = \frac{V_{d,j}^*}{v_0^*}, \quad \nu_{i,j} = \frac{v_{i,j}^*}{v_0^*}, \quad z = \frac{z^*}{L^*}, \]
\[ \lambda = \frac{\lambda^*}{L^*}, \quad \theta = \frac{\theta^*}{L^*}, \quad \rho_m = \frac{\rho_m^*}{\rho_0^*}, \]
\[ h_m = \frac{h_m^*}{\Delta h_{\theta g}^*}, \quad P = \frac{P*}{\rho_f^* \theta^*}, \quad \theta_{\infty} = \frac{\theta_{\infty}}{\Delta \theta^*}, \]
\[ N_n = \frac{N_n^*}{\rho_f^*}, \quad N_t = \frac{N_t^*}{\rho_f^*}, \]
\[ N_{sub} = \frac{N_{sub}^*}{\rho_f^*}, \quad N_{	ext{pcb}} = \frac{N_{	ext{pcb}}^*}{\Delta h_{\theta g}^* \rho_f^* \rho_f^*}, \]
Appendix B. The integrals that appear in the final ordinary differential equation for the inlet velocity, \( v_i(t) \) (eq. (20))

\[
\begin{align*}
I_\ell & = I_1 + (C_0 - 1) I'_2 + \bar{V}_m I''_2,
\end{align*}
\]

where

\[
I'_2 = I_2(v_i(t-t'-\nu) = v_i(t-t')),
\]

\[
I''_2 = I_2(v_i(t-t'-\nu) = 1)
\]

and

\[
I_1 = \int_0^t v_i(t-t'-\nu) \, dt',
\]

\[
I_2 = \int_0^t e^{N_p \bar{c} / \rho_m(t')} v_i(t-t'-\nu) \, dt',
\]

\[
I_3 = \int_0^t e^{N_p \bar{c} / \rho_m(t')} \int_0^t e^{N_p \bar{c} / \rho_m(t')} \times v_i(t-t'-\nu) \, dt'' \, dt',
\]

\[
I_4 = \int_0^t e^{N_p \bar{c} / \rho_m(t')} \int_0^t e^{N_p \bar{c} / \rho_m(t')} \times v_i(t-t'-\nu) \, dt'' \, dt',
\]

\[
I_5 = \int_0^t v_i(t-t'-\nu) \int_0^t e^{N_p \bar{c} / \rho_m(t')} v_i(t-t'-\nu) \, dt'' \, dt',
\]

\[
I_6 = \int_0^t e^{N_p \bar{c} / \rho_m(t')} j^2(t, t') v_i(t-t'-\nu) \, dt',
\]

\[
I_7 = \int_0^t e^{N_p \bar{c} / \rho_m(t')} j^2(t, t') v_i(t-t'-\nu) \, dt',
\]

\[
I_8 = \int_0^s G(t') j^2(t, t') v_i(t-t'-\nu) \, dt',
\]

\[
I_9 = \int_0^s F(t') v_i(t-t'-\nu) \, dt',
\]

\[
I_{10} = \int_0^s F(t') j (t, t') v_i(t-t'-\nu) \, dt',
\]

\[
I_{11} = \int_0^s G(t') v_i(t-t'-\nu) \, dt',
\]

\[
I_{12} = \int_0^s G(t') j (t, t') v_i(t-t'-\nu) \, dt',
\]

\[
I_{13} = \int_0^s e^{N_p \bar{c} / \rho_m(t')} j (t, t') v_i(t-t'-\nu) \, dt',
\]

\[
I_{14} = \int_0^s G(t') j (t, t') v_i(t-t'-\nu) \, dt',
\]

where

\[
F(t') = e^{N_p \bar{c} / \rho_m(t')} \frac{d}{dt'} \left[ \log \rho_m(t') \right]
\]

and

\[
G(t') = \frac{e^{N_p \bar{c} / \rho_m(t')}}{\rho_m(t')}
\]

Nomenclature

- \( A \) cross sectional flow area
- \( C \) cubic order terms
- \( C_0 \) void distribution parameter
- \( D \) diameter
- \( Fr \) Froude number = \( \nu_0^* / g^* l^* \)
- \( N_t \) friction number
- \( N_p, N_{p_e}, N_{p_{eh}} \) subcooling number
- \( N_{p}, N_{p_{eh}} \) phase change number
- \( P \) pressure
- \( V_{\text{gi}}, V_{\text{g}}, V_{\text{g}} \) drift velocity = \( \nu_0 - j \)
- \( f \) friction factor
- \( g \) gravitational constant
- \( h \) enthalpy
- \( i \) \( \sqrt{-1} \)
- \( j \) volumetric flow rate
- \( k \) \( \alpha_0 \) + \( 1 - \alpha \) \( \nu_i \)
- \( l \) channel length
- \( q' \) wall heat flux
- \( s \) Laplace transform variable
- \( v \) velocity
- \( x \) quality
- \( \Gamma' \) \( \rho_v \sqrt{A^* h_g^*} \)
- \( \Delta \rho \) \( \rho_l - \rho_g \)
- \( \alpha \) void fraction
- \( \epsilon \) oscillation amplitude
- \( \delta \) expanded time of Lindstedt–Poincaré analysis
- \( \lambda \) position of the boiling boundary
- \( \mu \) small parameter of Lindstedt–Poincaré analysis
- \( \nu \) \( N_{\text{sub}} / N_{\text{p_{eh}}} \)
- \( \rho \) density
- \( \bar{c} \) exit
- \( f \) liquid
- \( g \) vapor
- \( i \) inlet
- \( m \) mixture

Subscripts

- \( c \) exit
- \( f \) liquid
- \( g \) vapor
- \( i \) inlet
- \( m \) mixture
14

Rizwan-uddin, J.J. Dorning / Nonlinear dynamics of a heated channel

0 boiling boundary
1 single phase
2 two phase

Special symbol
~ steady state

References

[16] The discrepancy between the dimensional and dimensionless parameters was pointed out to us by Prof. R.P. Roy.
[20] This was brought to our attention by Prof. R.T. Lahey.