ANALYTICAL SOLUTION OF TIME-DEPENDENT MULTILAYER HEAT CONDUCTION PROBLEMS FOR NUCLEAR APPLICATIONS

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ABSTRACT
Analytical solutions for one-dimensional time dependent multilayer heat conduction problems were developed several decades ago. Mathematical theory for such problems in more than one dimensions was also developed during that time. Several of these methods were based on separation of variable and finite integral transform. However, the application of these methods was hindered by the fact that the eigenvalue problems, which are essential for this methodology are difficult to solve. Moreover, in two and three dimensional Cartesian coordinates these eigenvalues were imaginary rendering their solutions even more difficult. It has been recently shown that similar problems in two dimensional cylindrical and spherical coordinates do not have imaginary eigenvalues. It is also helpful that the softwares which are capable of analytical manipulations are now ubiquitous. This paper discusses the methodology as well as possible application in nuclear reactors of analytical solutions of two-dimensional multilayer heat conduction in spherical and cylindrical coordinates.

1. INTRODUCTION
Multilayer components are of significant interest due to the added advantage of combining different thermophysical properties. These components find a wide range of applications in various automotive, space, chemical, civil and nuclear industries. Time dependent temperature distribution in such components, with the presence of sources (with all three types of boundary conditions) can be obtained analytically or numerically. Though not always available, exact analytical solutions are desirable since: (1) better insight can be gained through the mathematical form of an analytical solution compared to a discrete numerical solution; and (2) these analytical solutions can be used as benchmark or reference results to verify numerical algorithms and codes.

There are several approaches for solving such problems [1-3]. In this paper, only solutions obtained with separation of variables are discussed. Series Solutions of one-dimensional problems, using separation of variables, were obtained several decades ago [4]. However, since computation of eigenvalues needed for the solution, is difficult, these solution were of little use for obtaining temperature distribution. Only recently with advances in computational capabilities such distributions have been obtained. For the multidimensional cases a solution and generalized coordinates has been given by Yener and Ozisik (1974) [5]. In theory this solution is applicable to all types of multiregion (including multilayer) problems. However, practical implementation is not trivially straightforward. It has been found that in multilayer multidimensional Cartesian coordinates eigenvalues of the problem may be imaginary making the solution of the characteristic equation very difficult, if not impossible. In contrast it has been recently shown that in r-θ cylindrical and 2D and 3D spherical coordinates similar problems have real eigenvalues.
In the conventional nuclear reactors, heat conduction in the fuel rods is through several layers and is also asymmetric. Moreover, in pebble bed reactor, which is a new design proposed for the reactors, similar multilayer heat conduction problem exists in spherical coordinates. In this paper recently developed analytical solution in multilayer cylindrical and spherical coordinates and its applicability to the nuclear engineering problems is discussed. Development and application of such 1D problems is also discussed.

2. MATHEMATICAL FORMULATION FOR 1D PROBLEMS

Consider an $n$-layer one dimensional composite slab $(0 \leq r \leq r_n)$. All the layers are assumed to be isotropic in thermal properties and are in perfect thermal contact. Let $k_i$ and $\alpha_i$ be the temperature independent thermal conductivity and thermal diffusivity of the $i^{th}$ layer. Initially, at $t = 0$, the $i^{th}$ layer is at a specified temperature $(T_i(r, t) = f_i(r))$. For $t > 0$, all three kinds of boundary conditions are applicable to the inner ($i = 1$, $r = r_0$) and the outer ($i = n$, $r = r_n$) surfaces. In addition, time independent heat sources $g_i(r)$ are switched on in each layer at $t = 0$.

Under these assumptions, the governing heat conduction equation, along with the boundary and initial conditions, are as follows:

Governing equation:

$$\frac{1}{\alpha_i} \frac{\partial T_i}{\partial t}(r, t) = \frac{1}{r^p} \frac{\partial}{\partial r}\left( r^p \frac{\partial T_i}{\partial r}(r, t) \right) + \frac{g_i(r)}{k_i},$$

\[ r_{i-1} \leq r \leq r_i, 1 \leq i \leq n \]  

In the above equation,

$p = 0$ for Cartesian coordinates

$p = 1$ for cylindrical coordinates

$p = 2$ for spherical coordinates

Boundary conditions:

- Inner surface of $i^{th}$ layer ($i = 1$)

$$A_{in} \frac{\partial T_{i-1}}{\partial r}(r_0, t) + B_{in} T_i(r_0, t) = C_{in}$$  \(2\)

- Outer surface of $n^{th}$ layer ($i = n$)

$$A_{out} \frac{\partial T_n}{\partial r}(r_n, t) + B_{out} T_n(r_n, t) = C_{out}$$  \(3\)

- Inner interface of $i^{th}$ layer ($i \neq 1$)

$$T_i(r_{i-1}, t) = T_{i-1}(r_{i-1}, t)$$  \(4\)

$$k_i \frac{\partial T_i}{\partial r}(r_{i-1}, t) = k_{i-1} \frac{\partial T_{i-1}}{\partial r}(r_{i-1}, t)$$  \(5\)

- Outer interface of $i^{th}$ layer ($i \neq n$)

$$T_i(r_{i}, t) = T_{i+1}(r_{i}, t)$$  \(6\)

$$k_i \frac{\partial T_i}{\partial r}(r_{i}, t) = k_{i+1} \frac{\partial T_{i+1}}{\partial r}(r_{i}, t)$$  \(7\)

Initial condition:

$$T_i(r, t = 0) = f_i(r)$$  \(8\)

It is to be noted that boundary conditions either of the first, second or third kind can be imposed at $r = r_0$ and $r = r_n$ by choosing the coefficients in equations (2) and (3) appropriately. Furthermore, multiple layers with zero inner radius ($r_0 = 0$) can be simulated by assigning zero values to constants $B_{in}$ and $C_{in}$ in equation (2).

3. SOLUTION METHODOLOGY

In order to apply the separation of variables method, which is only applicable to homogenous problems, the non-homogenous problem has to be split into: 1) homogenous transient problem, and 2) non-homogenous steady state problem.

3.1. Homogeneous transient problem

Introduce $\bar{T}_i(r, t) \equiv T_i(r, t) - T_{ss,i}(r)$ in order to homogenize the problem, where $T_{ss,i}(r)$ is the corresponding steady state solution. Subsequently, homogenized equations corresponding to equations (1)-(10) are as follows:

Governing equation:
1 \frac{\partial \bar{T}_i}{\partial t}(r,t) = \frac{1}{r^p} \frac{\partial}{\partial r} \left(r^p \frac{\partial \bar{T}_i}{\partial r}(r,t)\right), \quad (9)

Initial condition:

\bar{T}_i(r,t = 0) = f_i(r) - T_{ss,i}(r) \quad (10)

### 3.2. Separation of Variables

Substituting the product form for temperature \( \bar{T}_i(r,t) \),

\[ \bar{T}_i(r,t) = R_i(r)\Gamma_i(t) \quad (11) \]

in equation (9), and then applying separation of variables, yield the following ODEs:

\[ \frac{1}{r} \frac{d}{dr} \left(r \frac{dR_i}{dr}\right) + \lambda^2_i R_i = 0 \quad (12) \]

\[ \frac{1}{\alpha_i} \frac{d\Gamma_i}{dt} + \lambda^2 \Gamma_i = 0 \quad (13) \]

where \( \lambda_i \) are constants of separation.

### 3.3. General Solution

In view of equations (11), (12) and (13), a general solution for equation (9) may be considered as:

\[ \bar{T}_i(r,t) = \sum_{m=1}^{\infty} D_m e^{-\alpha_i \lambda_m^2 r} R_m(\lambda_m r) \quad (14) \]

For \( \bar{T}_i(r,t) \) to be continuous at the layer interfaces, for all values of \( t \),

\[ \lambda_m = \lambda_m \sqrt{\frac{\alpha_i}{\alpha_i}} \quad (15) \]

In equation (14), \( R_m(\lambda_m r) \) are eigenfunctions corresponding to eigenvalue problem in the \( r \)-direction and are given by:

\[ R_m(\lambda_m r) = a_m S(\lambda_m r) + b_m C(\lambda_m r) \quad (16) \]

In the above equation,

\[ S(\lambda_m r) = \begin{cases} \sin(\lambda_m r) & \text{for } p = 0 \\ J_0(\lambda_m r) & \text{for } p = 1 \\ \sin(\lambda_m r)/r & \text{for } p = 2 \end{cases} \quad (17) \]

Orthogonality condition for the \( r \)-direction eigenfunctions, which is similar to that in , is:

\[ \sum_{m=1}^{\infty} \frac{k_i}{\alpha_i} \int_{r_i}^{r} r^p R_m(\lambda_m r) R_{m'}(\lambda_{m'} r) dr = \begin{cases} 0 & \text{if } m \neq l \\ N_m & \text{if } m = l \end{cases} \quad (18) \]

The eigenvalues and the coefficients \( a_m \) and \( b_m \) can be found using boundary conditions. Coefficients \( D_m \) can be obtained using initial condition and orthogonality condition.

### 4. 2D MULTILAYER PROBLEMS IN CYLINDRICAL AND SPHERICAL COORDINATES

The dependence in the \( \theta \) direction arises from the boundary conditions or the source term. In such a case one needs to solve the 2D problem. Same is true for the spherical coordinates. Details of the multilayer 2D heat conduction problem are given in [6-8]. A brief review is given in following subsections.

#### 4.1. Mathematical formulation in cylindrical coordinates

A schematic of the problem in cylindrical coordinates is given in Fig. 1. Following is the mathematical formulation for the problem.
Governing equation:
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_i(r, \theta, t)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T_i(r, \theta, t)}{\partial \theta^2} + \frac{g_i(r, \theta, t)}{k_i}, \quad r_{i-1} \leq r \leq r_i
\]  
, \quad (19)

Boundary conditions:
- Inner surface of 1st layer \((i = 1)\)
  \[
  A_{in} \frac{\partial T_i}{\partial r}(r_0, \theta, t) + B_{in} T_i(r_0, \theta, t) = C_{in}(\theta, t)
  \]  
, \quad (20)
- Outer surface of 1st layer \((i = n)\)
  \[
  A_{out} \frac{\partial T_n}{\partial r}(r_n, \theta, t) + B_{out} T_n(r_n, \theta, t) = C_{out}(\theta, t)
  \]  
, \quad (21)

4.2. Eigenfunction expansion in \(\theta\)-direction

Due to periodic boundary conditions in \(\theta\)-direction, \(T_i(r, \theta, t)\) can be expanded into angular eigenfunctions \((\cos(q \theta), \sin(q \theta)\) and a constant) as follows:

\[
T_i(r, \theta, t) = T_{i0}(r, t) + \sum_{q=1}^{\infty} T_{iq}(r, t) \cos(q \theta)
\]  
, \quad (25)

\[
+ \sum_{q=1}^{\infty} T_{iq}(r, t) \sin(q \theta)
\]

Source term \(g\) can also be expanded in a Fourier series similarly. The coefficient of that series can be obtained appropriately. Similarly, right-hand sides in the boundary conditions also be expanded. Fourier expansions are then substituted in governing equation, boundary conditions and initial condition resulting in following equations for each \(m\).

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_{iq}^\theta(r, t)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T_{iq}^\theta(r, t)}{\partial \theta^2} + \frac{g_{iq}^\theta(r, t)}{k_i}, \quad r_{i-1} \leq r \leq r_i
\]  
, \quad (26)

Boundary conditions:
- Inner surface of 1st layer \((i = 1)\)
  \[
  A_{in} \frac{\partial T_{iq}^\theta(r_0, t)}{\partial r} + B_{in} T_{iq}^\theta(r_0, t) = C_{in,q}(t)
  \]  
, \quad (27)
- Outer surface of 1st layer \((i = n)\)
  \[
  A_{out} \frac{\partial T_{iq}^\theta(r_n, t)}{\partial r} + B_{out} T_{iq}^\theta(r_n, t) = C_{out,q}(t)
  \]  
, \quad (28)
- Interface of \(i\)th layer \((i = 2, ..., n)\)
  \[
  T_{iq}^\theta(r_{i-1}, t) = T_{i-1,q}^{\theta}(r_{i-1}, t)
  \]  
, \quad (29)
- Initial condition:
  \[
  T_{iq}^\theta(r, \theta, t = 0) = f_{iq}^\theta(r)
  \]  
, \quad (31)
Note that Eq. (26) is similar to the 1D equation except the second term in the RHS of Eq. (10) is not present in the 1D case.

Solutions then can be found as in the 1D case for each $m$ and then substituted in Eq. (25). Only difference is that the general solution is now:

$$R_{iqm}(\lambda_{iqm} r) = a_{iqm} J_q(\lambda_{iqm} r) + b_{iqm} Y_q(\lambda_{iqm} r)$$  \hspace{1cm} (32)

4.3. Mathematical formulation in spherical coordinates

In the spherical coordinates following equations are used.

$$\frac{1}{\alpha} \frac{\partial T}{\partial t}(r, \theta, t) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r}(r, \theta, t) \right)$$

$$+ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta}(r, \theta, t) \right) + \frac{g_i(r, \theta)}{k_i}$$  \hspace{1cm} (33)

Since the maximum range of $\theta$ is from $0$ to $\pi$, the corresponding range of $\mu$, $-1 \leq \mu \leq 1$, (where $\mu = \cos \theta$) covers all the range in the $\theta$-domain.

Boundary conditions:

- Inner surface of $i$th layer ($i = 1$)

$$A_{in} \frac{\partial T}{\partial r}(r_0, \mu, t) + B_{in} T_i(r_0, \mu, t) = C_{in}(\mu)$$  \hspace{1cm} (34)

- Outer surface of $n$th layer ($i = n$)

$$A_{out} \frac{\partial T}{\partial r}(r_n, \mu, t) + B_{out} T_n(r_n, \mu, t) = C_{out}(\mu)$$  \hspace{1cm} (35)

- $\theta = \Psi$ surface ($i = 1, 2, ..., n$)

$$T_i(r, \mu, t) = 0 \text{ or } \frac{\partial T_i}{\partial \mu}(r, \mu = \mu_q, t) = 0,$$

$$\mu_q = \cos \Psi, \Psi < \pi$$  \hspace{1cm} (36)

- Interface of the $i$th layer ($i = 2, ..., n$)

$$T_i(r_{i-1}, \mu, t) = T_{i-1}(r_{i-1}, \mu, t)$$  \hspace{1cm} (37)

$$k_i \frac{\partial T_i}{\partial r}(r_{i-1}, \mu, t) = k_{i-1} \frac{\partial T_{i-1}}{\partial r}(r_{i-1}, \mu, t)$$  \hspace{1cm} (38)

Initial condition:

$$T_i(r, \mu, t = 0) = f_i(r, \mu)$$  \hspace{1cm} (39)

The solution methodology is similar to the cylindrical coordinates except that the eigenfunction expansion is carried in Legendre polynomials and the resulting general solution of the equation is as follows:

$$R_{iqm}(\lambda_{iqm} r) = a_{iqm} J_{q+0.5}(\lambda_{iqm} r)$$

$$\frac{\sqrt{r}}{} + b_{iqm} Y_{q+0.5}(\lambda_{iqm} r)$$  \hspace{1cm} (40)

5. APPLICATIONS IN NUCLEAR ENGINEERING

A nuclear fuel rod is shown in the Fig. 2. For the safety and design calculations one needs to know that heat conduction in these rods. 1D or 2D as appropriate multilayer heat conduction problem can be solved in cylindrical coordinates using the procedure described earlier in this paper.
Figure 3 A fuel element of the Pebble bed modular reactor

In the Fig. 3, a spherical fuel element for a pebble bed reactor is shown. The analytical solution for the spherical coordinates for multilayer problems can be used for the heat conduction analysis in this element. The methodology summarized in this paper can be used either for the computations themselves or validating the computer codes.

6. CONCLUSIONS

The analytical methods for time-dependent multilayer heat conduction in two-dimensional problems have been recently developed. These methods and their one-dimensional counterparts in one-dimension are useful for solving heat conduction problems in cylindrical as well as spherical coordinates. The methods discussed in this paper are based on separation of variables and finite integral transform.

The analytical methods applicable in cylindrical coordinates can be used for heat conduction analysis nuclear fuel rods in the existing as well as planned nuclear reactor designs. Similar methods in spherical coordinates have applications in some specific reactor designs such as pebble bed nuclear reactors.

7. REFERENCES